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Wealth Maximization and the Kelly Criterion

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A basic goal of individual and institutional investors is to maximize wealth over time. Achieving this goal requires the identification of opportunities where the investor has an edge and the proper allocation of capital to these opportunities. The focus in this commentary is on the capital allocation component of investing: what is the best way to allocate capital across asset classes, or even within an asset class, to maximize wealth over the long run? Let's consider the following coin toss game: the bettor tosses a fair coin four times, getting back six times the amount of the bet for heads and nothing for tails. What should be the size of the bet to maximize gains at the end of the game? We'll get to the answer in a moment.

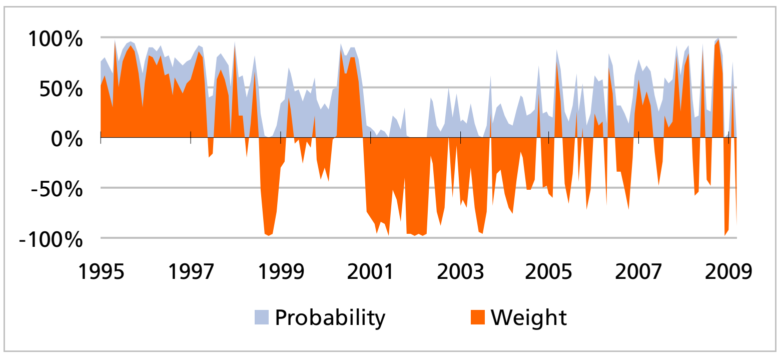
The standard answer to the asset allocation problem in finance is the mean/variance solution, based on the work of Harry Markowitz. Under the assumption of normally distributed returns, all the investor needs to know to fully characterize the distribution of returns is the portfolio's mean and variance. The mean is the *arithmetic,* or holding period, return of the portfolio. The outcome of the mean/variance optimization process is the efficient frontier, which is a set of portfolios that, for any given expected return, have the minimum level of risk (as measured by variance) among all portfolios with the same expected return. Given, then, an investor's risk preferences, expressed in the form of a utility function that likes return and dislikes risk, the optimal portfolio is that one among all efficient portfolios that maximizes the investor's utility function.

Unfortunately, though, wealth maximization over time is a problem mean/variance is not designed to address. Mean/variance optimization is *static*, in that it assumes a single period investment problem. Wealth maximization over time is a *dynamic*, multi-period problem. Focusing on arithmetic return is appropriate when there is no reinvestment issue, as in a single period. In dynamic problems that involve reinvestment, the *geometric mean* might be the appropriate quantity to maximize.[1] The mean variance approach does not allow for reinvestment and for deviations from normal distributions (skewness, kurtosis etc). Under these circumstances, there is an alternative portfolio construction process, known as the Kelly Criterion,[2] explicitly designed to maximize geometric mean return. The Kelly formula gives the bet size as the ratio of edge/odds.[3] Edge is the expected value of the bet and odds the gain/loss ratio.

Returning to our coin toss game and starting with a $100 bankroll, the expected value (edge) of the bet is 50% × ($5) + 50% × (-$1) = $2, while the odds are 5 to 1 because the gain per dollar bet is $5. Hence edge/odds is 2/5 and the optimal bet = 40%. The Kelly rule implies the bet size should be $40 and by doing so the bettor maximizes the geometric mean. Since the Kelly formula gives the bet size that maximizes the geometric mean, then for bets smaller than the optimal size, money is left on the table. Conversely for bets greater than optimal size, ruin is almost certain. In a fair game (as under market efficiency), edge is zero (prices follow a random walk) and the optimal bet size is zero.[4]

We have applied the Kelly criterion to construct optimal allocations in our developed vs. emerging markets asset allocation strategy. That strategy relies on a binomial logistic regression model that predicts the probability that developed markets will outperform emerging markets in any given month. The process evaluates only the direction of out-performance, not the magnitude. The model takes into account the following seven factors: lagged return differential, dividend yield differential, U.S. investment grade and high yield spreads, the level of the U.S. yield curve, commodity price changes, and the rate of change in the U.S. dollar index. Our model currently strongly favors emerging markets.

We compare two portfolio construction methods. A portfolio constructed using mean/variance optimization using our model's probability in place of expected return vs. a portfolio constructed on the basis of the Kelly criterion, where the probability of winning is the probability of out-performance from our logistic regression model. The Kelly-based portfolio is shown below. When the probability of developed market outperformance is high, the allocation to developed markets is high. When that probability is close to 50% the allocation to developed (or emerging) markets is close to zero, and in fact it is identically zero for odds of 1:1. When the probability of develop market out-performance is less than 50%, the allocation shifts in favor of emerging markets.



It's important to emphasize that the Kelly criterion is designed to maximize the *growth rate* of the portfolio (and avoid bankruptcy in the short run) rather than maximize the *information ratio*. As it turns out, in this case the Kelly-based strategy outperforms the mean variance strategy in a number of ways: cumulative return 111% vs. 60% over 171 months; information ratio is 0.64 vs. 0.40; annualized down deviation is 4.9% vs. 5.6%, and Sortino ratio (return-to-down deviation ratio) is 1.10 vs. 0.60.

	Kelly	Mean Variance
Number of months	171	171
Cumulative Return	111%	60%
Annualized	5.4%	3.4%
Annl. Std. Dev.	8.5%	8.5%
IR	0.64	0.40
Profitable Months	58%	58%
Max	10.1%	7.8%
Min	-6.3%	-8.3%
AnnlDownDev	4.9%	5.6%
Sortino	1.10	0.60
1 yr Ret.	10.31%	-0.56%
3 yr Ret.	2.77%	0.36%
5 yr Ret.	3.03%	0.27%

In summary, information edge is the key element of investing. If the probability of out-performance from our model was not a reliable signal, sizing the allocations based on it would not improve results. Second, the allocation size should be a function of expected opportunity, as is the case for both the Kelly method and mean/variance. Third, mean/variance is not designed to maximize long-term wealth assuming reinvestment and non-normally distributed returns while the Kelly rule is. ■

1 Paul Samuelson has argued against this on the basis that geometric mean maximization could violate the utility function. "The Fallacy of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling," Proceedings of the National Academy of Sciences, vol. 68, 10, October 1971, 2493-2496

2 Kelly, J.L., Jr., "A New Interpretation of Information Rate," Bell System Technical Journal, 1956, 917-926

3 Poundstone, William, "Fortune's Formula: The Untold Story of the Unscientific Betting System That Beat The Casinos and Wall Street," (New York: Hill and Wang, 2005)

4 The origin of this idea comes from Bell Labs scientist Claude Shannon and his work on information theory.